

# Gödel-type universes in $f(T)$ gravity

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## Abstract

The issue of causality in  $f(T)$  gravity is investigated by examining the possibility of existence of the closed timelike curves in the Gödel-type metric. By assuming a perfect fluid as the matter source, we find that the fluid must have an equation of state parameter greater than minus one in order to allow the Gödel solutions to exist, and furthermore the critical radius  $r_c$ , beyond which the causality is broken down, is finite and it depends on both matter and gravity. Remarkably, for certain  $f(T)$  models, the perfect fluid that allows the Gödel-type solutions can even be normal matter, such as pressureless matter or radiation. However, if the matter source is a special scalar field rather than a perfect fluid, then  $r_c \rightarrow \infty$  and the causality violation is thus avoided.

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## I. INTRODUCTION

General relativity (GR) is established in the framework of the Levi-Civita connection, therefore there is only curvature rather than torsion in the spacetime. On the other hand, one can also introduce other connections, such as the Weitzenböck connection, into the same spacetime where only torsion is reserved. Thus, there is no such a thing as curvature or torsion of spacetime, but only curvature or torsion of connection. Basing on the Weitzenböck connection, Einstein [1] introduced firstly the Teleparallel Gravity (TG) in his endeavor to unify gravity and electromagnetism with the introduction of a tetrad field. TG can, as is well known, show up as a theory completely equivalent to GR since the difference between their actions (the actions of TG and GR are the torsion scalar  $T$  and Ricci scalar  $R$ , respectively) is just a derivative term [2–7].

Recently, a modification of TG, called  $f(T)$  theory [8–66], has spurred an increasing deal of attention, as it can explain the present accelerated cosmic expansion discovered from observations (the Type Ia supernova [67, 68], the cosmic microwave background radiation [69, 70], and the large scale structure [71, 72], etc.) without the need of dark energy.  $f(T)$  theory is obtained by generalizing the action  $T$  of TG to an arbitrary function  $f$  of  $T$ , which is very analogous to  $f(R)$  theory (see [73–77] for recent review) where the action  $R$  of GR is generalized to be  $f(R)$ . An advantage of  $f(T)$  theory is that its field equation is only second order, while in  $f(R)$  gravity it is forth order.

It has been found that  $f(T)$  theory can give an inflation without an inflaton [9, 10], avoid the big bang singularity problem in the standard cosmological model [11], realize the crossing of phantom divide line for the effective equation of state [13, 14], and yield an usual early cosmic evolution [17, 18]. But, at the same, this theory lacks the local Lorentz invariance [61, 62], and this results in the appearance of extra degrees of freedom [63], the broken down of the first law of black hole thermodynamic [64], and the problem in cosmic large scale structure [65].

In this paper, we plan to study the causality issue of  $f(T)$  theory by examining the possibility of existence of the closed timelike curves in the Gödel spacetime [78]. The Gödel metric is the first cosmological solution with rotating matter to the Einstein equation in

GR. Since the Gödel solution is very convenient for studying whether the closed timelike curves exist, it has been used widely to test the causality issue. For example, Gödel found that the closed timelike solution cannot be excluded in GR, assuming a cosmological constant or a perfect fluid with its pressure equal to the energy density. Gödel's work has been generalized to include other matter sources, such as, the vector field [79–81], scalar field [85–87], spinor field [89–94] and tachyon field [95]. In addition, the Gödel-type universes [82–84, 94] have also been studied in the framework of other theories of gravitation, such as TG [96],  $f(R)$  gravity [97–99] and string-inspired gravitational theory [100, 101].

Here, assuming that the matter source is the perfect fluid or a scalar field, we aim to find out the condition for non-violation of causality in  $f(T)$  gravity. The paper is organized as follows. We give, in Sec. II, a brief review of  $f(T)$  theory and the vierbein of a general cylindrical symmetry metric in Sec. III. The Gödel-type universe in  $f(T)$  theory is discussed in Sec. IV. With an assumption of different matter sources, we investigate the issue of causality in Sec. V. Finally, we present our conclusions in Sec. VI.

## II. $f(T)$ GRAVITY

In this section, we give a brief view of  $f(T)$  gravity. We use the Greek alphabet ( $\mu, \nu, \dots = 0, 1, 2, 3$ ) to denote tensor indices, that is, indices related to spacetime, and middle part of the Latin alphabet ( $i, j, \dots = 0, 1, 2, 3$ ) to denote tangent space (local Lorentzian) indices. TG, instead of using the metric tensor, uses tetrad,  $e_\mu^i$  or  $e_i^\mu$  (frame or coframe), as the dynamical object. The relation between frame and coframe is

$$e_i^\mu e_\mu^j = \delta_i^j, \quad e_i^\mu e_\nu^i = \delta_\nu^\mu, \quad (1)$$

and the relation between tetrad and metric tensor is

$$g_{\mu\nu} = e_\mu^i e_\nu^j \eta_{ij}, \quad \eta_{ij} = e_i^\mu e_j^\nu g_{\mu\nu}, \quad (2)$$

where  $\eta_{ij} = \text{diag}(1, -1, -1, -1)$  is the Minkowski metric.

Different from GR, the Weitzenböck connection is used in TG

$$\Gamma_{\mu\nu}^\lambda = e_i^\lambda \partial_\nu e_\mu^i = -e_\mu^i \partial_\nu e_i^\lambda. \quad (3)$$

As a result, the covariant derivative, denoted by  $D_\mu$ , satisfies:

$$D_\mu e_\nu^i = \partial_\mu e_\nu^i - \Gamma_{\nu\mu}^\lambda e_\lambda^i = 0 . \quad (4)$$

To describe the difference between Weitzenböck and Levi-Civita connections, a contorsion tensor  $K^\rho_{\mu\nu}$  needs to be introduced:

$$K^\rho_{\mu\nu} \equiv \Gamma_{\mu\nu}^\rho - \overset{\circ}{\Gamma}_{\mu\nu}^\rho = \frac{1}{2}(T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu}) . \quad (5)$$

Here  $T^\rho_{\mu\nu}$  is the torsion tensor

$$T^\rho_{\mu\nu} = \Gamma_{\nu\mu}^\rho - \Gamma_{\mu\nu}^\rho = e_i^\rho(\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) , \quad (6)$$

and  $\overset{\circ}{\Gamma}_{\mu\nu}^\rho$  denotes the Levi-Civita connection

$$\overset{\circ}{\Gamma}_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) . \quad (7)$$

By defining the super-potential  $S_\sigma{}^{\mu\nu}$

$$S_\sigma{}^{\mu\nu} \equiv K^{\mu\nu}{}_\sigma + \delta_\sigma^\mu T^{\alpha\nu}{}_\alpha - \delta_\sigma^\nu T^{\alpha\mu}{}_\alpha , \quad (8)$$

we obtain the torsion scalar  $T$

$$T \equiv \frac{1}{2}S_\sigma{}^{\mu\nu}T^\sigma{}_{\mu\nu} = \frac{1}{4}T^{\alpha\mu\nu}T_{\alpha\mu\nu} + \frac{1}{2}T^{\alpha\mu\nu}T_{\nu\mu\alpha} - T_{\alpha\mu}{}^\alpha T^{\nu\mu}{}_\nu . \quad (9)$$

In TG, the Lagrangian density is given by:

$$L_T = \frac{eT}{2\kappa^2} , \quad (10)$$

where,  $e = \det(e_\mu^i) = \sqrt{-g}$ ,  $\kappa^2 \equiv 8\pi G$ . Generalizing  $T$  to be an arbitrary function  $f$  of  $T$  in the above expression, we obtain the Lagrangian density of  $f(T)$  theory

$$L_T = \frac{ef(T)}{2\kappa^2} . \quad (11)$$

Adding a matter Lagrangian density  $L_M$  to Eq. (11), and varying the action with respect to the vierbein, one finds the following field equation of  $f(T)$  theory:

$$[e^{-1}\partial_\mu(ee_i^\rho S_\rho{}^{\nu\mu}) - e_i^\lambda S^{\rho\mu\nu}T_{\rho\mu\lambda}]f_T(T) + e_i^\rho S_\rho{}^{\nu\mu}\partial_\mu(T)f_{TT}(T) + \frac{1}{2}e_i^\nu f(T) = \kappa^2 e_i^\rho T^{\mu\nu}{}_\rho . \quad (12)$$

Here  $f_T = df(T)/dT$ ,  $f_{TT} = d^2f(T)/dT^2$ , and  $T^\nu_\rho$  is the matter energy-momentum tensor. In a coordinate system, this field equation can be rewritten as

$$A_{\mu\nu}f_T(T) + S_{\nu\mu}{}^\sigma(\nabla_\sigma T)f_{TT}(T) + \frac{1}{2}g_{\mu\nu}f(T) = \kappa^2 T^{\text{em}}_{\mu\nu}, \quad (13)$$

where

$$\begin{aligned} A_{\mu\nu} &= g_{\sigma\mu}e_\nu^i[e^{-1}\partial_\xi(ee_i^\rho S_\rho{}^{\sigma\xi}) - e_i^\lambda S^{\rho\xi\sigma}T_{\rho\xi\lambda}] \\ &= G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = -\nabla^\sigma S_{\nu\sigma\mu} - S^{\rho\lambda}{}_\mu K_{\lambda\rho\nu}, \end{aligned} \quad (14)$$

$G_{\mu\nu}$  is the Einstein tensor, and  $\nabla_\sigma$  is the covariant derivative associated with the Levi-Civita connection. The trace of Eq. (12) or (13), which can be used to simplify and constrain the field equation, can be expressed as

$$- [2e^{-1}\partial_\sigma(eT_\rho{}^{\rho\sigma}) + T]f_T(T) + S_\rho{}^{\rho\sigma}(\partial_\sigma T)f_{TT}(T) + 2f(T) = \kappa^2 T^{\text{em}}, \quad (15)$$

where,  $T^{\text{em}} = T^\mu{}_\mu = g^{\mu\nu}T_{\mu\nu}$  is the trace of the energy-momentum tensor. Clearly, in the case of TG,  $f(T) = T$ , and Eq. (15) reduces to

$$T - 2e^{-1}\partial_\sigma(eT_\rho{}^{\rho\sigma}) = \kappa^2 T^{\text{em}}, \quad (16)$$

which shows an equivalence between GR and TG since

$$-R = T - 2e^{-1}\partial_\sigma(eT_\rho{}^{\rho\sigma}). \quad (17)$$

### III. VIERBEIN FOR CYLINDRICAL SYMMETRY METRIC

Since the Gödel-type metric is usually expressed in cylindrical coordinates  $[(r, \phi, z)]$ , we consider a general cylindrical symmetry metric

$$ds^2 = dt^2 + 2H(r)dtd\phi - dr^2 - G(r)d\phi^2 - dz^2, \quad (18)$$

where  $H$  and  $G$  are the arbitrary functions of  $r$ . This metric can be re-expressed in the following form

$$ds^2 = [dt + H(r)d\phi]^2 - D^2(r)d\phi^2 - dr^2 - dz^2, \quad (19)$$

where

$$D(r) = \sqrt{G(r) + H^2(r)} . \quad (20)$$

Since the local Lorentz invariance is violated in  $f(T)$  theory and the vierbein have six degrees of freedom more than the metric, one should be careful in choosing a physically reasonable tetrad in terms of Eq.(2). Here, we choose the tetrad anstaz of the cylindrical symmetry metric to be:

$$e^i_\mu \equiv \begin{pmatrix} 1 & 0 & H & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad e^\mu_i \equiv \begin{pmatrix} 1 & 0 & -\frac{H}{D} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{D} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \quad (21)$$

Using Eqs. (3–9), one can find that the Weitzenböck invariant  $T$  is

$$T = \frac{1}{2} \left( \frac{H'}{D} \right)^2 , \quad (22)$$

where a prime presents a derivative with respect to  $r$ .

Substituting the vierbein given in Eq. (21) into Eq. (13), we obtain the following non-zero components of the  $f(T)$  field equation:

$$\nu = 0, i = 0$$

$$\left( T - \frac{D''}{D} + \frac{HT'}{2H'} \right) f_T(T) + \left( \frac{HT}{H'} - \frac{D'}{D} \right) T' f_{TT}(T) + \frac{1}{2} f(T) = \kappa^2 T^{em}_0 \quad (23)$$

$$\nu = 0, i = 2$$

$$\left( HT + \frac{T'D^2}{2H'} \right) f_T(T) + \frac{T'H'}{2} f_{TT}(T) - \frac{H}{2} f(T) = \kappa^2 \left( T^{em}_2 - HT^{em}_0 \right) , \quad (24)$$

$$\nu = 1, i = 1$$

$$- T f_T(T) + \frac{1}{2} f(T) = \kappa^2 T^{em}_1 , \quad (25)$$

$$\nu = 2, i = 0$$

$$T' \left[ \frac{1}{2H'} f_T(T) + \sqrt{\frac{T}{2}} f_{TT}(T) \right] = \kappa^2 T^{em}_2 , \quad (26)$$

$$\nu = 2, i = 2$$

$$-T f_T(T) + \frac{1}{2} f(T) = \kappa^2 \left( T^2_2 - H T^2_0 \right), \quad (27)$$

$$\nu = 3, i = 3$$

$$-\frac{D''}{D} f_T(T) - \frac{T' D'}{D} T' f_{TT}(T) + \frac{1}{2} f(T) = \kappa^2 T^3_3. \quad (28)$$

Apparently, the non-symmetric components of the modified Einstein equation are consistent with the tetrad ansatz given in Eq. (21). In the above equations, all other components of  $T^{\mu}_{\nu}$  must be zero, which means that,  $T^{\mu}_{\nu}$ , has the cylindrical symmetry as expected. In a Gödel-type spacetime, the energy-momentum tensor in a local basis,  $T^{em}_{ab}$  given in (42), has a general form:  $T^{em}_{ab} = \text{diag}(\rho, p_1, p_2, p_3)$ . Using  $T^{em}_{\mu\nu} = e^a_{\mu} e^b_{\nu} T^{em}_{ab}$ , we have

$$T^{em}_{00} = \rho, \quad T^{em}_{11} = p_1, \quad T^{em}_{22} = H^2 \rho + D^2 p_2, \quad T^{em}_{33} = p_3, \quad T^{em}_{02} = T^{em}_{20} = H \rho. \quad (29)$$

One can then find easily

$$T^{em2}_0 = 0, \quad T^{em0}_2 = H \left( T^{em0}_0 - T^{em2}_2 \right). \quad (30)$$

Thus, Eq. (24) seems to give an extra constraint on  $f(T)$  gravity. This equation is satisfied automatically in a Gödel-type spacetime, since  $T$ , as shown in Eq. (34), is a constant in a Gödel-type universe. Furthermore, it is easy to see that, in a Gödel-type spacetime, Eq. (24) gives the same expression as Eq. (27). Four independent field equations are obtained, which is consistent with the ansatz of tetrad. In addition, one can check that the field equations (23-28) for the vierbein given in (21) can also be obtained from an action constructed by replacing the specific form of  $T$  (22) with the general action of  $f(T)$  theory. Therefore, the dynamical equations are consistent, which means that the tetrad ansatz given in Eq. (21) is a good guess for the Gödel-type spacetime.

#### IV. GÖDEL-TYPE UNIVERSE IN $f(T)$ THEORY

To show the possibility of existence of the closed timelike curves and the causality feature in  $f(T)$  gravity, we consider the Gödel-type metric, which has the form of Eq. (18)

with  $H$  and  $G$  being:

$$H(r) = \frac{4\omega}{m^2} \sinh^2 \left( \frac{mr}{2} \right), \quad (31)$$

$$G(r) = \frac{4}{m^2} \sinh^4 \left( \frac{mr}{2} \right) \left[ \coth^2 \left( \frac{mr}{2} \right) - \frac{4\omega^2}{m^2} \right], \quad (32)$$

where  $\omega$  and  $m$  ( $-\infty < m^2 < +\infty, 0 < \omega^2$ ) are two constant parameters used to classify different Gödel-type geometries. Thus, we have

$$D(r) = \frac{1}{m} \sinh(mr). \quad (33)$$

Substituting the expressions of  $H$  and  $D$  into Eq. (22), one can obtain easily

$$T = 2\omega^2, \quad (34)$$

which is a positive constant.

If  $G(r) < 0$ , Eq. (18) shows that one type of closed timelike curve, called noncausal Gödel circle [78], exists in the case of  $t, z, r = \text{const.}$  This means a violation of causality. For a particular case of  $0 < m^2 < 4\omega^2$ , the causality violation region, i.e.,  $G(r) < 0$  region, exists if

$$\tanh^2 \frac{mr}{2} < \frac{m^2}{4\omega^2}. \quad (35)$$

Thus, one can define a critical radius  $r_c$  [78, 97–99]

$$\tanh^2 \frac{mr_c}{2} = \frac{m^2}{4\omega^2}, \quad (36)$$

beyond which,  $G(r) < 0$  and causality is violated. When  $m = 0$ , the critical radius is  $r_c = 1/\omega$ . When  $m^2 = 4\omega^2$ ,  $r_c = +\infty$ , which means that a breakdown of causality is avoided. Thus, the codomain range of  $r_c$  is  $r_c \in (1/\omega, +\infty)$ . Therefore, the condition for non-violation of causality is  $m^2 \geq 4\omega^2$  or  $r < r_c$ . For the case in which  $m^2 = -\mu^2 < 0$ , both  $H(r) = \frac{4\omega}{\mu^2} \sin^2(\frac{\mu r}{2})$  and  $G(r) = \frac{4}{\mu^2} \sin^4(\frac{\mu r}{2}) [\cot^2(\frac{\mu r}{2}) - \frac{4\omega^2}{\mu^2}]$  are periodic functions. Thus, an infinite circulation of causal and noncausal ranges appears [98, 99].

It is easy to see that, if one further defines a set of bases  $\{\theta^a\}$ :

$$\theta^0 = dt + H(r)d\phi, \quad \theta^1 = dr, \quad (37)$$



$$\theta^2 = D(r)d\phi, \quad \theta^3 = dz, \quad (38)$$

the Goödel-type line element can be simplified to be:

$$ds^2 = \eta_{ab}\theta^a\theta^b, \quad (39)$$

where  $\eta_{ab} = \text{diag}(1, -1, -1, -1)$  is the Minkowski metric. By choosing  $\{\theta^a\}$  as basis, the  $f(T)$  field equation (13) becomes:

$$A_{ab}f_T(T) + \frac{1}{2}\eta_{ab}f(T) = \kappa^2 T_{ab}^{em}. \quad (40)$$

Here, both  $f(T)$  and  $f_T(T)$  are evaluated at  $T = 2\omega^2$ . The second term of Eq. (13) is discarded in obtaining the above equation since the torsion scalar  $T$  is a constant. We find that the nonzero components of  $A_{ab}$  are

$$A_{00} = 2\omega^2 - m^2, \quad A_{11} = A_{22} = 2\omega^2, \quad A_{33} = m^2. \quad (41)$$

Thus, we obtain a very simple form of the field equation in  $f(T)$  gravity, which will help us discuss the causality issue.

## V. CAUSALITY PROBLEM IN $f(T)$ THEORY

One can see, from Eq. (40), that, in order to discuss the causality problem, the matter source is a very important component. As was obtained in [97–99], different matter sources may lead to different results. In this paper, we assume that the matter source consists of two different components: a perfect fluid and a scalar field. Thus, the energy-momentum tensor  $T_{ab}^{em}$  has the form

$$T_{ab}^{em} = T_{ab}^m + T_{ab}^s, \quad (42)$$

where,  $T_{ab}^m$  and  $T_{ab}^s$  correspond to the energy-momentum tensors of the perfect-fluid and the scalar field, respectively. In basis  $\{\theta^a\}$ ,  $T_{ab}^m$  and  $T_{ab}^s$  can be expressed as

$$T_{ab}^m = (\rho + p)u_a u_b - p\eta_{ab}, \quad (43)$$

$$T_{ab}^s = D_a \Phi D_b \Phi - \frac{1}{2}\eta_{ab} D_c \Phi D_d \Phi \eta^{cd}, \quad (44)$$

where  $u_a = (1, 0, 0, 0)$ ,  $\rho$  and  $p$  are the energy density and pressure of the perfect fluid, respectively, and  $p = w\rho$  with  $w$  being the equation of state parameter.  $\Phi$  is the scalar field, and  $D_a$  denotes the covariant derivative relative to the local basis  $\theta^a$ . The scalar field equation is  $\square\Phi = \eta^{ab}\nabla_a\nabla_b\Phi = 0$ . It is easy to prove that  $\Phi(z) = \varepsilon z + \text{const}$  with a constant amplitude  $\varepsilon$  satisfies this field equation [94]. Using the solution  $\Phi(z) = \varepsilon z + \text{const}$ , one can obtain the nonvanishing components of  $\overset{s}{T}_{ab}$

$$\overset{s}{T}_{00} = -\overset{s}{T}_{11} = -\overset{s}{T}_{22} = \overset{s}{T}_{33} = \frac{\varepsilon^2}{2}, \quad (45)$$

Thus, the energy-momentum tensor of matter source becomes

$$\overset{em}{T}_{ab} = \text{diag}\left(\rho + \frac{\varepsilon^2}{2}, w\rho - \frac{\varepsilon^2}{2}, w\rho - \frac{\varepsilon^2}{2}, w\rho + \frac{\varepsilon^2}{2}\right). \quad (46)$$

Substituting Eqs. (41) and (46) into the  $f(T)$  field equation (Eq. (40)), we find

$$(2\omega^2 - m^2)f_T(T) + \frac{1}{2}f(T) = \kappa^2\left(\rho + \frac{\varepsilon^2}{2}\right); \quad (47)$$

$$2\omega^2 f_T(T) - \frac{1}{2}f(T) = \kappa^2\left(w\rho - \frac{\varepsilon^2}{2}\right); \quad (48)$$

$$m^2 f_T(T) - \frac{1}{2}f(T) = \kappa^2\left(w\rho + \frac{\varepsilon^2}{2}\right). \quad (49)$$

Since the effective Newton gravity constant in  $f(T)$  gravity becomes  $G_{N,eff} = G_N/f_T(T)$  [66], only the case  $f_T(T) > 0$  will be considered in the following in order to ensure a positive  $G_{N,eff}$ . From Eqs. (47) and (48), one can derive a relation between  $m$  and  $\omega$ :

$$m^2 = 2\omega^2 \left[ 1 + \frac{\varepsilon^2}{\rho(1+w) + \varepsilon^2} \right], \quad (50)$$

which implies that the critical radius of the Gödel's circle, Eq. (36), satisfies

$$\tanh^2\left(\frac{mr_c}{2}\right) = 1 - \frac{\rho(1+w)}{2[\rho(1+w) + \varepsilon^2]}. \quad (51)$$

Obviously, different matter sources give rise to different critical radii and therefore different causality structures, e.g. when  $\varepsilon \rightarrow 0$ , we have a finite  $r_c$ , while for  $\rho \rightarrow 0$ ,  $r_c = \infty$ .

Therefore, a violation of causality may occur for the case of a perfect fluid as the matter source, whereas causality is preserved in the case of a scalar field. In order to show the causality feature in more detail and the conditions for obtaining the Gödel-type solutions, we will divide our discussion into two special cases:  $\varepsilon \rightarrow 0$  and  $\rho \rightarrow 0$ . In addition, a concrete  $f(T)$  model will be considered.

**A.**  $\varepsilon^2 \rightarrow 0$

$\varepsilon^2 \rightarrow 0$  corresponds to the case that the universe only contains a perfect fluid. Since  $f_T(T) > 0$ , Eqs. (47), (48), and (49) reduce to:

$$m^2 = 2\omega^2 ; \quad (52)$$

$$T f_T(T) = \kappa^2 \rho (1 + w) ; \quad (53)$$

$$f(T) = 2\kappa^2 \rho . \quad (54)$$

From Eqs. (53, 54), it is easy to see that, in the limit of general relativity without a cosmological constant ( $f(T) = T$ ),  $w = 1$  is required to ensure the existence of the Gödel-type solutions [88, 97–99]. This means that a violation of causality in general relativity is only possible for the so-called stiff fluid ( $w = 1$ ) which is not a normal fluid in our Universe. In  $f(T)$  theory,  $T f_T(T) > 0$  and  $\rho > 0$  lead to  $w > -1$ . So, the perfect fluid must satisfy the weak energy condition ( $\rho > 0$  and  $\rho(1 + w) > 0$ ). Using the above results, the equation of state can be expressed as a function of the torsion scalar:

$$w = \frac{2T f_T(T)}{f(T)} - 1 . \quad (55)$$

Different from general relativity that requires  $w = 1$  for perfect-fluid Gödel solutions, the equation of state parameter of the fluid  $w$  in  $f(T)$  gravity can differ from one and its value is determined by concrete  $f(T)$  models. For example, a special  $f(T) = \lambda T^\delta$  gives  $w = 2\delta - 1$ , from which one can see that  $w$  can be an arbitrary number for an arbitrary  $\delta$ . So, even normal matter, such as pressureless matter or radiation, can lead to a violation

of causality in certain  $f(T)$  theories. This indicates that the issue of causality violation seems more severe in  $f(T)$  gravity than in general relativity where only an exotic stiff fluid allows the existence of Gödel-type solutions. From Eqs. (52), (53) and (54), and using  $T = 2\omega^2$ , we find that the critical radius given in Eq. (51) becomes

$$r_c = 2 \tanh^{-1} \left( \frac{1}{\sqrt{2}} \right) \cdot \sqrt{\frac{f_T(T)}{(1+w)\kappa^2\rho}}, \quad (56)$$

which is dependent both on the specifics of  $f(T)$  theory and the properties of the perfect fluid.

Now, let us consider a concrete power law  $f(T)$  model [20]

$$f(T) = T - \alpha T_* \left( \frac{T}{T_*} \right)^n, \quad (57)$$

where  $\alpha$  and  $n$  are model parameters, and  $T_*$  is a special value of the torsion scalar, which is introduced to make  $\alpha$  dimensionless.  $|n| \ll 1$  is required in order to obtain an usual early cosmic evolution [17]. The current cosmic observations give that  $\alpha = -0.79^{+0.35}_{-0.79}$  and  $n = 0.04^{+0.22}_{-0.33}$  at the 68.3% confidence level [15]. Thus, a negative  $\alpha$  is favored by observations. In term of Eq. (55), the equation of state of the perfect fluid becomes

$$w = 1 - \frac{2\alpha(n-1)T_*^{1-n}}{T^{1-n} - \alpha T^{1-n}}. \quad (58)$$

The equation above can be re-expressed as

$$\frac{\alpha(2n-1-w)}{1-w} = \left( \frac{T}{T_*} \right)^{1-n} > 0, \quad (59)$$

where a positive  $T/T_*$  is considered. Recalling  $\alpha < 0$  and  $w > -1$ , from Eq. (59) one can obtain the possible ranges of  $w$  for the Gödel-type universes

$$1 > w > -1 + 2n \quad (1 > n > 0), \quad 1 > w > -1 \quad (n < 0). \quad (60)$$

For this power law model, the critical radius has the form

$$r_c = 2 \left[ \frac{\alpha(2n-1-w)}{1-w} \right]^{\frac{1}{2(n-1)}} \tanh^{-1}(1/\sqrt{2}), \quad (61)$$

which is determined completely by the model parameters and the equation of state of the perfect fluid.

**B.**  $\rho \rightarrow 0$

This is the case of a scalar field as the matter source. Eqs. (47), (48), and (49) now reduce to

$$m^2 = 4\omega^2 , \quad (62)$$

$$T f_T(T) = \kappa^2 \varepsilon^2 , \quad (63)$$

$$f(T) = 3\kappa^2 \varepsilon^2 . \quad (64)$$

Note that (63) and (64) combined together admit a relation between  $T$  and  $f(T)$ :

$$3T f_T(T) - f(T) = 0 , \quad (65)$$

which constrains the class of solutions with no violation of causality. For the power law model, the causal Gödel-type solution gives that the torsion scalar should satisfy

$$T = 2\omega^2 = \left[ -\frac{(1-3n)\alpha}{2} \right]^{\frac{1}{1-n}} T_* . \quad (66)$$

Thus,  $n < 1/3$  is required if the numerator of  $\frac{1}{1-n}$  is not even since the observations show  $\alpha < 0$ .

## VI. CONCLUSIONS

$f(T)$  theory, a new modified gravity, provides an alternative way to explain the present accelerated cosmic acceleration with no need of dark energy. Some problems, including large scale structure, local Lorentz invariance, and so on, of this modified gravity have been discussed. In this paper, we study the issue of causality in  $f(T)$  theory by examining the possibility of existence of the closed timelike curves in the Gödel metric. Assuming that the matter source is a scalar field or a perfect fluid, we examine the existence of the Gödel-type solutions. For the scalar field case, we find that  $f(T)$  gravity allows a particular Gödel-type solution with  $r_c \rightarrow \infty$ , where  $r_c$  is the critical radius beyond which

the causality is broken down. Thus, the violation of causality can be forbidden. In the case of a perfect fluid as the matter source, we find that the fluid must have an equation of state parameter greater than minus one and this parameter should satisfy Eq. (55) for the Gödel-type solutions to exist. For certain  $f(T)$  models, the perfect fluid that allows the Gödel-type solutions can even be normal matter, such as pressureless matter or radiation. Since the critical radius  $r_c$  of perfect fluid Gödel-type solutions which depends on both matter and gravity is finite, the issue of causality violation seems more severe in  $f(T)$  gravity than in general relativity where only an exotic stiff fluid allows the existence of Gödel-type solutions.

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